Function Theory of a Complex Variable (E2): Exercise sheet 2

1. Show that $f(z) = |z|^2$ has a derivative only at the origin.

2. Consider the function $f(z) = \sqrt{|x||y|}$. Show this satisfies the Cauchy-Riemann equations at the origin, but is not differentiable there.

3. Find the most general harmonic polynomial of the form $ax^3 + bx^2y + cxy^2 + dy^3$. Determine the conjugate harmonic function, and the corresponding analytic function.

4. Let D be the unit disc. Assume it is made of a heat conducting material that only loses heat through its boundary. At steady state, the temperature in the disc $T(x, y)$ is a harmonic function. Suppose on the boundary the temperature satisfies:

$$T(\cos\theta, \sin\theta) = 2\sin^2\theta.$$

   (a) By finding an appropriate analytic function $f(z)$, determine the temperature $T(x, y)$ within the disc. *NB. You may assume that the harmonic function T is determined uniquely by the given boundary conditions, and moreover find it helpful to rewrite $2\sin^2\theta$ as $1 - \cos(2\theta)$.*

   (b) Find equations for and sketch the isothermals (lines of constant temperature).

   (c) Assuming that heat flows perpendicular to the isothermals, find equations for and sketch the heat flow lines.

5. (a) Expand $(1 - z)^{-m}$, m a positive integer, in powers of z.

   (b) Expand $(1 + z)^{-1}$, in powers of $z - 1$. What is the radius of convergence (about 1)?

6. Supposing $\sum_n a_n z^n$ has radius of convergence R, determine the radius of convergence of:

   (a) $\sum_n a_n z^{2n}$;

   (b) $\sum_n a_n^2 z^n$.

7. Find the real and imaginary parts of:

   (a) e^{e^z} for $z \in \mathbb{C}$;

   (b) z^z for $z \in \mathbb{C}\backslash\{0\}$.